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Combinatorial volume preserving flows and taut foliations

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Abstract. A π_1-injective closed surface in an orientable 3-manifold with a tangentially smooth, transversely C^0 taut foliation can be homotoped to an immersed surface which is either transverse to the foliation except at isolated saddle tangencies or mapped into a leaf.

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§0. Introduction

In 1979 two remarkable results were published in minimal surface theory. D. Sullivan [Su] showed that given a C^2-taut foliation in a closed orientable 3-manifold M, there exists a Riemannian metric on M such that each leaf is a minimal surface. R. Schoen and S. T. Yau [SY] showed that if T is a closed orientable surface and $f : T \to M$ is π_1-injective, then f is homotopic to a least area immersion. Consequently [H1], if T is a connected orientable π_1-injective surface in the manifold M with C^2-taut foliation $\mathcal{F}$, then after homotopy either T is an immersed surface mapping into a leaf of $\mathcal{F}$ or T is transverse to $\mathcal{F}$ except at isolated saddle tangencies.

Our main result generalizes this last result to immersed incompressible surfaces in taut C^0-foliations.

Theorem 2.7. *If T is a connected closed orientable incompressible surface in the closed orientable 3-manifold M with taut foliation $\mathcal{F}$, then after homotopy either T is an immersed surface mapped into a leaf of $\mathcal{F}$ or T is an immersed surface transverse to F except at isolated saddle tangencies.*

Remark. Actually the theorem is true in greater generality, e.g. all the orientability hypothesis can be dropped. See Theorems 3.4 and 3.7.

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Around 1970 R. Roussarie [Ro] and W. Thurston [Th] had independently obtained Theorem 2.7 under the additional hypothesis that T is embedded in M, via a proof that generalized Alexander's proof of the PL Schoenflies theorem. Basically center tangencies are canceled with saddle tangencies in the course of isotoping T to the desired form. In 1981 [G1] this result was obtained for immersed incompressible surfaces in finite depth taut foliations. In the immersed setting pushing down a center (in an attempt to cancel a saddle) may create branched points. This gave rise to a very nasty situation which the author only resolved in the finite depth setting.

This paper abstracts the ideas of minimal surface theory, Haken's normal surface theory [Ha] and Thurston's proof [H1, appendix] of Sullivan's theorem to obtain an elementary and intuitive proof of Theorem 2.7.

We introduce the following

Definition 0.1. A *combinatorial volume preserving flow* (ϕ, Γ) on the 3-manifold M consists of a triangulation Γ and an assignment of a positive integer $\phi(\eta)$ and orientation to each 1-simplex η of Γ. Additionally, at each vertex the sum of the values of inward pointing edges equals the sum of the values of the outward pointing edges.

Here is the idea of the proof of Theorem 2.7. First we construct a combinatorial volume preserving flow transverse to $\mathcal{F}$. Such a flow gives rise to a weighted PL metric on a triangulation of M by assigning the weight at edge η to be $\phi(\eta)$. With respect to this PL metric each leaf of $\mathcal{F}$ is a least weight surface. Second, with respect to this weighted PL metric we homotop T to a normal least area surface. Third and finally we eliminate the center tangencies of $\mathcal{F}|T$ via center/saddle cancellation.

Amazingly after the initial preparation of Steps 1-2, this last step proceeds as in the embedded case, [Ro]. In fact our homotopy is simpler than the homotopies in the classical embedded case of [Ro] and [Th]. See Remark 2.11. Finally the homotopy of Step 3 is regular and in particular avoids the nastiness of §7 of [G1], for reasons described in Remark 2.10.

The main result of §1 is

Proposition 1.5. *Let $\mathcal{F}$ be a codimension-1 foliation on the closed orientable 3-manifold. $\mathcal{F}$ is taut if and only if M has a combinatorial volume preserving flow (ϕ, Γ) compatible with $\mathcal{F}$.*

In §2 we prove Theorem 2.7. In §3 we show how Theorem 2.7 can be generalized by dropping the orientability hypotheses and allowing for manifolds with boundary. We also discuss how Haken's theory of PL metrics can be generalized and state an important open problem in the subject.

See [§5 G3] for other problems in the theory of minimal surfaces and foliations.

Notation and Terminology. If E is a topological space, then $|E|$ denotes the number of components of E. A map $f : M \to N$ between manifolds is *proper* if $f^{-1}(\partial N) = \partial M$. If X is a triangulation or cell complex, then X^i denotes its i-skeleton. $\mathbb{N}$ denotes the positive integers.

§1. Combinatorial volume preserving flows on 3-manifolds

Definition 1.1. Let F be a 2-dimensional foliation in the closed orientable 3-manifold M. Recall that $\mathcal{F}$ is *taut* if $\mathcal{F}$ is transversely oriented and each leaf nontrivially intersects a closed transverse curve. (The latter condition is equivalent to the existence of some closed transverse curve intersecting each leaf.) We define a *nontransversely orientable taut foliation*, to be a nontransversely orientable foliation such that each leaf nontrivially intersects a closed transverse curve. We will assume that all leaves of $\mathcal{F}$ are smoothly embedded in M, although the transverse structure may only be C^0.

Recall that a *normal arc* in a 2-simplex is a properly embedded arc intersecting distinct edges and a *normal disc* in a 3-simplex is a properly embedded disc whose boundary is the union of 3 or 4 normal arcs. If T is a surface and M has the triangulation Γ, then $f : T \to M$ is *normal* if the preimage of each 3-simplex consists of discs which are individually mapped, via embeddings, to normal discs.

We say that the foliation $\mathcal{F}$ intersects the triangulation Γ *normally* if $\mathcal{F}$ is transverse to the 1 and 2-skeleta of Γ, distinct vertices of Γ lie in distinct leaves of $\mathcal{F}$ and for any leaf L and 3-simplex σ each non point component of $L \cap \sigma$ is a normal disc. To simplify the technical details of what follows all foliations in this paper will satisfy the following additional property.

(1.2) The closed star $St(v)$ of any vertex v lies in a foliation chart of $\mathcal{F}$.

Using the Reeb stability theorem [Re] it is a topological exercise to show that these conditions are equivalent to the following one. Vertices of Γ lie on distinct leaves of $\mathcal{F}$ and if v is a vertex of Γ, then there exists a foliation chart of $\mathcal{F}$ homeomorphic to $\mathbb{R}^2 \times \mathbb{R}$ such that $St(v) \subset \mathbb{R}^2 \times \mathbb{R}$ and each simplex of $St(v)$ embeds in $\mathbb{R}^2 \times \mathbb{R}$ as a geometric (straight) simplex in $\mathbb{R}^3 = \mathbb{R}^2 \times \mathbb{R}$. (Consequently each vertex of $St(v)$ lies at a distinct height.)

If the transversely oriented foliation $\mathcal{F}$ is normal to Γ, then the transverse orientation induces a linear ordering on the vertices $\{v_0, \cdots, v_3\}$ of each 3-simplex σ. So $v_i > v_j$ if the orientation induced from $\mathcal{F}$ on 1-simplex connecting v_i to v_j points from v_i to v_j.

We say that the combinatorial flow (ϕ, Γ) is *compatible* with $\mathcal{F}$ if $\mathcal{F}$ intersects Γ normally and the orientations induced on the edges of Γ by ϕ agree with the transverse orientation of $\mathcal{F}$.

Lemma 1.3. *If $\mathcal{F}$ is a codimension-1 foliation on the 3-manifold M, then $\mathcal{F}$ is normal to a smooth triangulation Γ on M.*

Proof. By J. H. C. Whitehead [Mu], any smooth manifold has a smooth triangulation Γ_1. By passing to subdivision and then perturbing slightly we obtain a triangulation Γ_2 such that for each 3-simplex σ of Γ_2, $\mathrm{St}(\sigma)$ lies in a foliation chart and each 1 and 2-simplex is transverse to $\mathcal{F}$ except at isolated points of tangency of a standard nature. Therefore $\mathcal{F}|\sigma$ has only finitely many leaves which either meet a vertex of σ or have a tangency with a 1 or 2 dimensional face of σ. Let σ' be the space obtained from σ by deleting its vertices and these exceptional leaves. Since σ lies in a chart, $\mathcal{F}|\sigma$ has no holonomy. Thus $\mathcal{F}|\sigma'$ is a disjoint union of a finite number of product foliations of the form $K \times (0,1)$, where K is a compact planar surface and furthermore $\Gamma_2^1 \cap (K \times (0,1)) = X \times (0,1)$ for some discrete set $X \subset \partial K$. It is now routine to subdivide Γ_2 to a triangulation Γ_3 such that $\mathcal{F}$ is normal to Γ_3 and satisfies (1.2). □

Remark 1.4. The analogue of Lemma 1.3 is true for codimension-1 foliations of any dimension, since Whitehead's theorem is dimension free.

Proposition 1.5. *Let $\mathcal{F}$ be a transversely oriented codimension-1 foliation on the closed orientable 3-manifold M. $\mathcal{F}$ is taut if and only if M has a combinatorial volume preserving flow (ϕ, Γ) compatible with $\mathcal{F}$.*

Proof. Assume that $\mathcal{F}$ is taut. Let Γ_1 be any smooth triangulation of M normal to $\mathcal{F}$. Since $\mathcal{F}$ is taut, for every 3-simplex σ there exists an oriented piecewise smooth path E_σ transverse to $\mathcal{F}$ from the bottom (i.e. minimal)vertex of σ to the top (i.e. maximal) vertex of σ such that the orientation on E_σ is induced from the transverse orientation on $\mathcal{F}$. One can choose these paths and find a subdivision Γ of Γ_1 such that each E_σ is an embedded path in Γ^1. Choose Γ to have the additional property that $\mathcal{F}$ is a normal with respect to it and satisfies (1.2).

For each 1-simplex η of Γ there exists a closed curve containing η lying in the 1-skeleton of Γ which is transverse to $\mathcal{F}$. In fact, if η is contained in a 3-simplex σ of Γ_1, then by extending the lowest endpoint of η down and the highest one up and staying within σ one obtains a path in $\Gamma^1 \cap \sigma$ from the highest vertex of σ to the lowest one which passes through η. By taking the union of this path with E_σ one obtains the desired closed curve. Now consider a finite collection $\mathcal{C} = \{\phi_1, \cdots, \phi_n\}$ of such curves whose union contains the entire 1-skeleton of Γ. If η is a 1-simplex define $\phi(\eta)$ to be the number of times $\mathcal{C}$ crosses η, counting with multiplicity. The orientation on $\mathcal{F}$ induces the orientation on η. By construction, for each vertex v of Γ, the values of inward pointing edges equals the values of outward pointing edges.

Conversely if (ϕ, Γ) is a combinatorial volume preserving flow, compatible with $\mathcal{F}$, then the orientations on edges of Γ induce a transverse orientation on $\mathcal{F}$. Since ϕ is volume preserving and $\phi(\eta) \in \mathbb{N}$ for each 1-simplex η, there exists oriented closed transversals $\phi_1, \cdots, \phi_n$ which induce the function ϕ as in the previous paragraph. Thus each leaf of $\mathcal{F}$ hits a closed transversal and hence $\mathcal{F}$ is taut. □

§2. Least Area Foliations

Definition 2.1. If T is a compact connected surface not equal to the 2-sphere or 2-disc, and M is a 3-manifold, then the proper map $f : T \to M$ is defined to be an *incompressible* surface if for each essential simple closed curve γ in T, $0 \neq [f_{\#}(\gamma)] \in \pi_1(M)$. If T is a 2-sphere, then f is incompressible if it is homotopically non trivial. If T is a 2-disc, then f is incompressible if T cannot be homotoped into ∂M rel ∂T. (See [He] for the classical definition of embedded incompressible surface.) Usually the map f is suppressed and one just refers to T being incompressible.

The proper map $f : T \to M$ is *boundary incompressible* if each essential properly embedded arc $\alpha \in T$ has the property that $f \mid \alpha$ cannot be homotoped rel $\partial\alpha$ into ∂M.

In the literature the word incompressible is sometimes used to mean π_1-injective; however, we believe that a distinction needs to be made. That is because certain results (e.g. Theorem 2.7) in 3-manifold theory hold more generally for incompressible surfaces and because of the questionable nature of the unresolved

Simple Loop Conjecture. 2.2. *Closed orientable incompressible surfaces in orientable 3-manifolds are π_1-injective.*

Remark 2.3. See [G2] for a proof of the 2-dimensional version and [H2], [RW] for some partial results in dimension 3.

Definition 2.4. Let $\phi : \{\text{1-simplices of } \Gamma\} \to (0, \infty)$ where Γ is a triangulation on M. If S is a compact surface and $f : S \to M$ is transverse to Γ^1, then define

$$w(S) = \sum_{\{\text{1-simplices } \eta \text{ of } \Gamma\}} \phi(\eta)|f^{-1}(\eta)|.$$

We call $w(S)$ the *weight* of S and ϕ a *PL-weight function.* In this paper, ϕ will arise from a combinatorial volume preserving flow and hence ϕ will have only positive integral values.

Lemma 2.5. *Let M be a closed orientable irreducible 3-manifold. Let S be a compact surface lying in a leaf of the taut foliation $\mathcal{F}$ and let (ϕ, Γ) be a combinatorial volume preserving flow dual to $\mathcal{F}$ such that $\partial S \cap \Gamma^1 = \emptyset$. Then $w(S) \leq w(T)$ if T is any immersed surface in M disjoint from Γ^0 and transverse to Γ^1 such that $\partial S = \partial T$ and $0 = [S \cup (-T)] \in H_2(M)$.*

Proof. Let Φ be the 1-cycle defined by (ϕ, Γ). Since intersection number is a homology invariant we obtain $w(T) \geq \Phi(T) = \Phi(S) = w(S)$. $\square$

Lemma 2.6. *Let Γ be a triangulation on the compact, orientable, irreducible 3-*

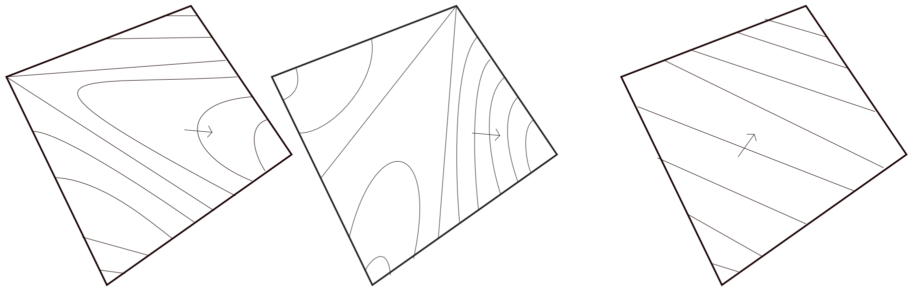
manifold and ϕ a PL-weight function on Γ. An incompressible, boundary incompressible surface T can be properly homotoped to a normal immersed surface of least weight.

Proof. Among all mappings $f : T \to M$ homotopic to the given one, let $g : T \to M$ be one of least weight. By the usual Haken procedure, we can assume that if α is a component of $g^{-1}(\kappa), \kappa$ a 2-simplex of Γ, then $g|\alpha$ is an embedding onto a normal arc. Also we can assume that these normal arcs intersect each other transversely and without triple points. Among all such maps let h be one which minimizes $(w(h), n_1(h), n_2(h))$ lexicographically, where $n_2(h)$ is the number of normal arcs and $n_1(h)$ is the number of double points among the normal arcs. By the minimality of h, irreducibility of M and incompressibility and boundary incompressibility of T, if D is a component of $h^{-1}(\sigma), \sigma$ a 3-simplex, then D is a disc. If for each edge e of $\sigma, |h^{-1}(e) \cap D| \leq 1$, then $h|\partial D$ is an embedding and hence we can assume (after a homotopy of D rel ∂D) that $h|D$ is an embedding onto a normal disc. If for some edge e, $|h^{-1}(e) \cap D| > 1$, then by minimality of $w(h)$ all intersections of ∂D with e occur with the same sign. In that case we can homotop $h|D$ rel ∂D to a map also called h, such that $h|D$ is a generic branched immersion into σ and some branched point in σ is connected via an embedded double curve to a point $x \in \partial\sigma$. If $x \in \partial M$, then the boundary incompressibility can be used to reduce $w(h)$. Otherwise, by doing the homotopy which corresponds to pushing the branched point (as in [G2]) out of σ one obtains a new map $k : T \to M$ with $w(k) = w(h)$. Either some component of $k^{-1}(\kappa)$, κ a 2-simplex, does not map to a normal curve, in which case $w(k)$ can be reduced, or $n_1(k) = n_1(h) - 1$ which again is a contradiction. □

Theorem 2.7. *If T is a connected closed orientable incompressible surface in the closed orientable 3-manifold M with taut foliation $\mathcal{F}$, then after homotopy either T is an immersed surface mapped into a leaf of $\mathcal{F}$ or T is an immersed surface transverse to F except at isolated saddle tangencies.*

Proof of Theorem 2.7. Let (ϕ, Γ) be a combinatorial volume preserving flow compatible with $\mathcal{F}$. By Lemma 2.6 we can assume that T is an immersed normal surface of least weight in its homotopy class. After this initial preparation, the proof of Theorem 2.7 now follows by applying the procedure of [Ro]. We discuss Roussarie's work in sufficient detail to check that his procedure, which describes an isotopy of an embedded surface, generalizes in the natural way to a regular homotopy of the immersed surface T. See Remarks 2.10.

Let Δ denote the cell structure on T induced by Γ. After a further homotopy we can assume that distinct vertices of Δ map to distinct leaves of $\mathcal{F}$, each edge of Δ maps to an arc transverse to $\mathcal{F}$, and that the 2-cells of Δ map to normal discs in 3-simplices of Γ transverse to $\mathcal{F}$. (Note that there are essentially two distinct ways to do this for quadrilateral normal discs (henceforth known as *quads*) whose



The possible types of induced foliations on quads.

Figure 2.1.

boundaries alternate going up and down.) See Figure 2.1. Thus the resulting immersion of T will be transverse to $\mathcal{F}$ except at local minimal, maxima and multi-saddles that occur at vertices of Δ. A map with the properties of the preceeding sentence will be called a Γ-*normal transverse immersion*.

The reader should enumerate all the possibilities for what neighborhoods of vertices of Δ look like inside of M.

Let $\mathcal{G}$ denote the singular foliation on T induced by $\mathcal{F}$. Singularities of index 1 are called *centers* and those of negative index are called *saddles*. If $\mathcal{G}$ has a center we will homotope our mapping to either an immersion into a leaf or to a new Γ-normal transverse immersion which has fewer centers. Thus by induction we can either homotope our mapping to either an immersion onto a leaf or a Γ-normal transverse immersion which has no centers. In the latter case a small homotopy will then change the mapping of T into a normal smooth immersion whose only tangencies with $\mathcal{F}$ are of saddle type, thereby completing the proof of Theorem 2.7.

By construction a center must correspond to a vertex w of Δ. We will assume that with respect to the transverse orientation on $\mathcal{F}$ it corresponds to a local maximum. By pushing down on T near w we create a little plateau P as in Figure 2.2. In general we define a *plateau* to be a subsurface of T which is mapped to a leaf of $\mathcal{F}$. In what follows, except for those isolated moments when a plateau P of T passes through Γ^0, Γ will induce a cellulation Δ on T such that 2-cells of Δ map to normal discs of Γ.

Claim 1. If P is a disc plateau of T, then T can be pushed down near P until one of the following events happens.

a) The plateau absorbs a vertex of Δ which is not a saddle point.

b) The plateau meets a saddle point at a vertex of Δ.

c) The plateau passes through a vertex of Γ.

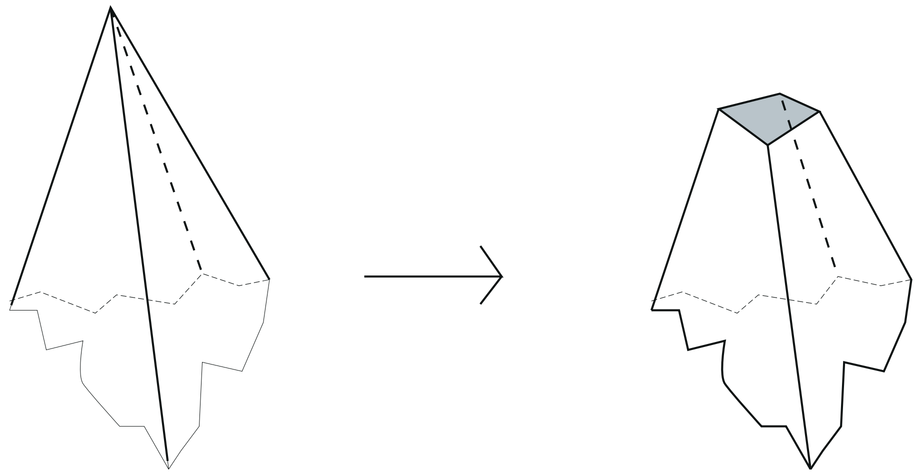


Figure 2.2.

Remark 2.8. In order to make the proof of this claim and the ones that follow transparent, the reader should at the appropriate moments completely understand what happens to a 2-cell J of Δ as viewed in a 3-simplex σ of Γ. E. g. consider what happens to J in the course of creating and pushing a plateau. At first J is normally embedded and transverse to $\mathcal{F}$. Then J gets pushed down near a vertex w which is a local maximum of J. Thus $J \cap P$ is a little triangle containing a vertex of Δ which we still call w. J can then get pushed down more and more until the little triangle evolves and encounters another vertex of Δ (case a) or b)) or encounters a vertex of Γ (case c). See Figure 2.3.

Now imagine that J is a normal disc which is transverse to $\mathcal{F}$ except along a subsurface $J \cap P$ which is mapped into a leaf of $\mathcal{F}$ and which is a local maximum of J. Up to topological conjugacy there are only a few possibilities which we leave to the reader to enumerate. As before by pushing J down near $J \cap P$ the plateau restricted to J will encounter another vertex of Δ (case a) or b)) or encounter a vertex of Γ (case c). This case includes the possibility that J is a triangle mapped into a leaf of $\mathcal{F}$ which suddenly vanishes as it is pushed out the bottom vertex of σ.

Proof of Claim 1. That P can be pushed at all follows from the Reeb Stability Theorem [Re]. Since the restriction of $\mathcal{F}$ to neighborhoods of 3-simplices are product foliations, Reeb's result appears particularly simple in our context. Indeed, if J is a 2-cell of Δ such that $J \cap P \neq \emptyset$, then Remark 2.8 describes how to push J until it encounters an event. Patching local pushes gives rise to a global push that includes an event. Figure 2.3 shows examples of events a), c) as restricted to a 2-cell of Δ. $\square$

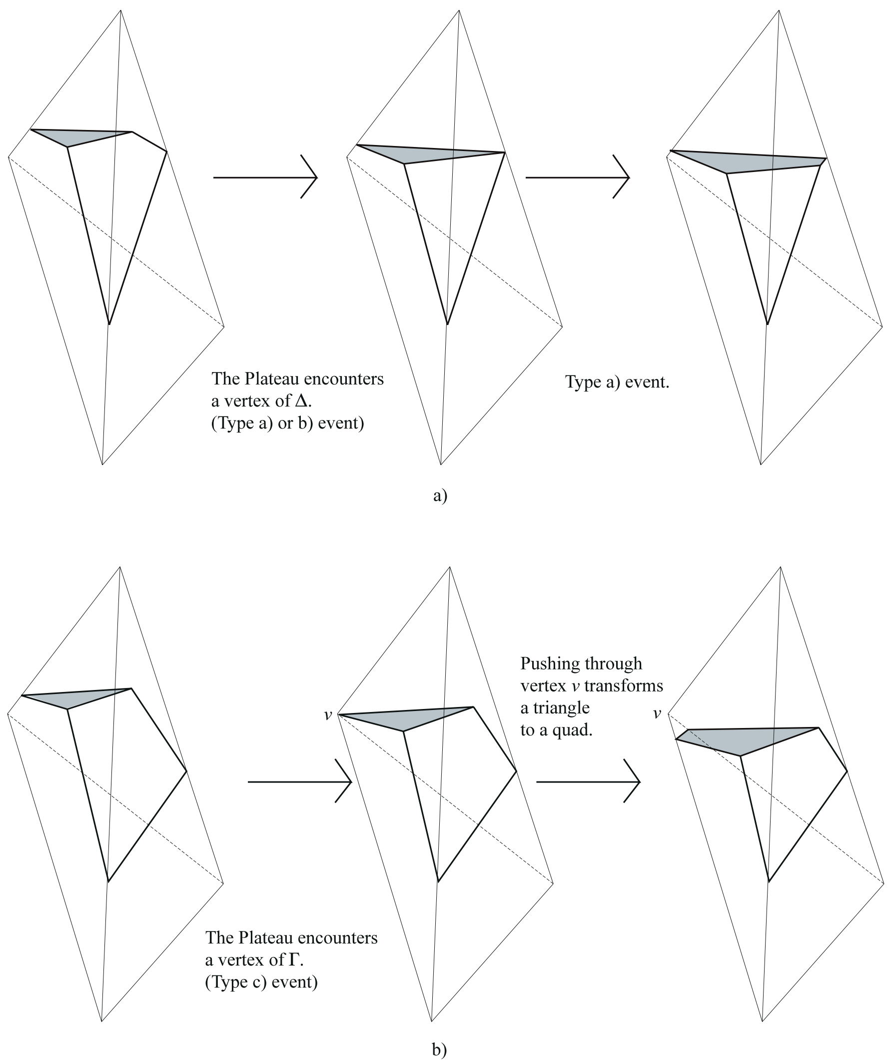


Figure 2.3.

Remark 2.9. One can recapture most of the essential information about how to homotop T given the data $T \cap \Gamma^2$ and $\mathcal{F} \mid \Gamma^2$. To obtain the complete story one must know which quad embedding was used for the alternating quads.

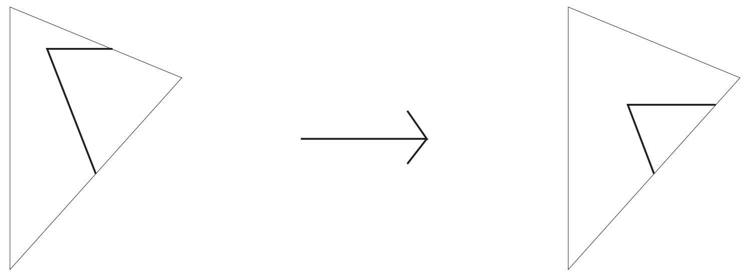


Figure 2.4.

Claim 2. If the plateau P encounters a type a) event, then a further small push creates a new plateau P_1 such that $\mid \Delta^0 - P_1 \mid = \mid \Delta^0 - P \mid -1$.

Proof of Claim 2. Use the fact that vertices of Δ which are not in P occur at distinct heights. $\square$

Claim 3. Pushing a plateau slightly past a vertex v of Γ, yields a new surface T_1 homotopic to T which is a least weight normal surface.

Proof of Claim 3. Since the weights on Γ^1 arise from a combinatorial volume preserving flow transverse to $\mathcal{F}, w(T) = w(T_1)$, and hence T_1 is a least weight surface. T_1 is normal else it would undergo a weight reducing move. See Figure 2.4 for a 2-dimensional version.

The changes in Δ as one crosses v are described as follows. There are three types of 3-simplices near v. Those called σ_m that have v as a minimum, those called σ_M that have v as a maximal point, and those called σ_n where v is neither maximal or minimal. (Maximal and minimal make sense since we are assuming that $St(v) \subset \mathbb{R}^2 \times \mathbb{R}$.) Just before crossing v, for each σ_m, P has a unique triangle very close to v which gets squashed out in the act of crossing v. For each σ_M, P has a triangle created which lies very close to v. See Figure 2.5

Finally a triangle in $P \cap \sigma_n$ gets transformed to a quad or conversely. The former (resp. latter) occurring if v is the second (resp. third) highest vertex in σ_n. See Figure 2.3b. As mentioned previously, what does not happen is that a 2-cell of Δ lying in a 3-simplex σ_n gets transformed into a non normal disc in σ_n. Otherwise T_1 and hence T is homotopic to a surface with smaller weight. $\square$

Claim 4. Events a) and c) happen only finitely many times before a type b) event happens.

Remark 2.10. i) Claims 1)-4) demonstrate the power of our least weight normal surface technique compared with the classical method of pushing down centers, as applied in §7 of [G1]. In that setting one may encounter neighborhoods of centers

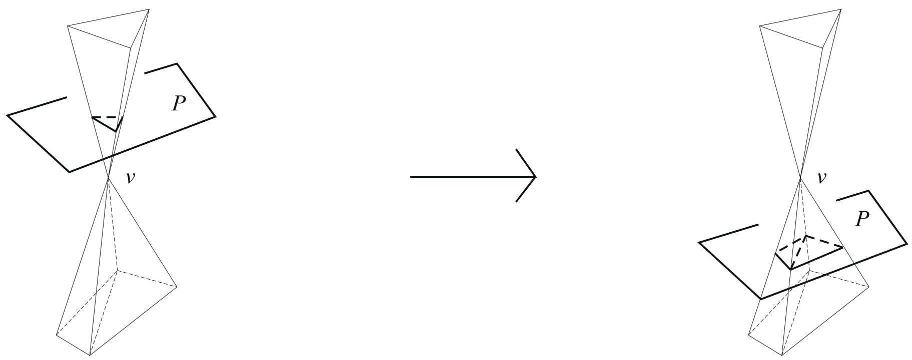


Figure 2.5.

which appear as in Figure 7.2 of [G1]. The act of pushing down the center creates a branched immersion as in Figure 7.3 of [G1]. In our setting such a neighborhood does not exist else it would give rise to a weight reducing move as discussed in Claim 3 and Figure 2.4.

ii) The homotopy from the surface T produced from Lemma 2.6 to the surface satisfying the conclusion of Theorem 2.7 is a regular homotopy.

Proof of Claim 4. Throughout the homotopy $w(T)$ remains fixed. On the other hand events of type a) increase the integer $w(P)$, while type c) events leave $w(P)$ unchanged. Since $w(P) \leq w(T), P$ can experience only finitely many type a) events. Now suppose that we have an infinite sequence of consecutive type c) events. Define $\Delta_0 = \Delta$, the cellulation at the beginning of our infinite sequence and let Δ_t denote the cellulation at time t. Here time is parametrized so that if n is a positive integer, then Δ_n denotes Δ at the moment of the n'th event of our sequence. If $0 < s,t < 1$, then there is a canonical correspondence between the vertices of Δ_{n+t} and those of Δ_{n+s}. In the passage from $\Delta_{n-\epsilon}$ to Δ_n to $\Delta_{n+\epsilon}$, $p \geq 1$ vertices get squashed to a single vertex which then blows up into $q \geq 1$ vertices. Meanwhile there is a cannonical bijection between the other vertices of $\Delta_{n-\epsilon}$ and those of $\Delta_{n+\epsilon}$. We will say that the vertex $w_t \in \Delta_t$ is stabilized if it is not involved in a type c) event at any time $s \geq t$. Such a vertex is canonically identified with a vertex $w_s \in \Delta_s$ for all $s > t$.

Let P_t (resp. T_t) denote the plateau P (resp. surface T) at time t. Let $G_t = \Delta_t \cap P_t$. Let w_0 denote a vertex of $\Delta_0 \cap P_0$ which lies on a 1-cell β_0 of the 1-skeleton of G_0 and connects w_0 to an endpoint of G_0. Suppose that w_0 is not stabilized. The free arc β_0 is canonically identified with a free arc β_t of G_t for all t, even if w_0 is involved in a type c) event at time n. See Figure 2.6. Thus $\beta_{n+\epsilon}$ picks out a newly created vertex which we call $w_{n+\epsilon}$. An inspection of the normal arc of $T_{n+\epsilon}$ which contains $\beta_{n+\epsilon}$ shows that $w_{n+\epsilon}$ is stabilized, for $w_{n+\epsilon}$ cannot experience a type c) event before it experiences a type b) or type a) event.

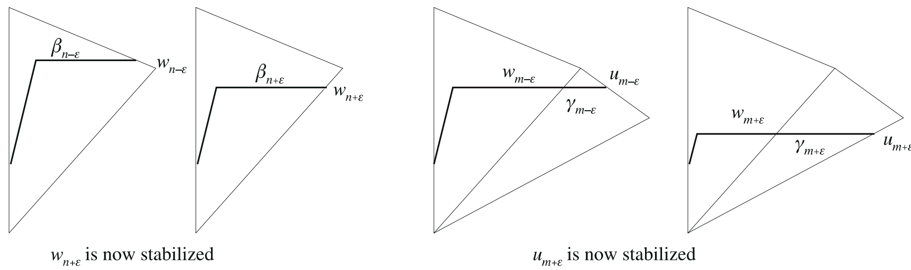


Figure 2.6.

Thus at some finite time t_1, every vertex of Δ_{t_1} which lies on a free edge of G_{t_1} is stabilized. Let E_1 be the set of those vertices of Δ_{t_1} which lie on a free edge of G_{t_1}.

If u_{t_1} is a vertex of Δ_{t_1} which is connected by an edge γ_{t_1} of G_{t_1} to an E_1 vertex then either u_{t_1} is stabilized or as in the previous paragraph, γ_{t_1} is cannonically identified with some edge γ_t of G_t for all time $t > t_1$ and u_{t_1} evolves to a stabilized vertex u_s which is connected to γ_s. Therefore at some time t_2 each element of the set E_2 is stabilized. Here E_2 is the set of vertices of G_{t_2} which are connected via an edge to an E_1 vertex. In a similar manner one defines $E_3, E_4, \cdots$. Each Δ_t has at most $w(T)$ vertices, since each vertex of Δ_t contributes a positive integral amount to $w(T_t) = w(T)$. Thus $E_i = \emptyset$ if i is sufficiently large. This contradicts the hypothesis that there exists an infinite sequence of consecutive type c) events. $\square$

Recall that $\mathcal{G}$ denotes the singular foliation induced on T by $\mathcal{F}$.

Claim 5. Center singularities of $\mathcal{G}$ can only appear as type (N, M), where $M \geq N - 1$ and $N, M \in \mathbb{N}$. This means that if p is a center of $\mathcal{G}$, then there is a closed neighborhood $U \subset T$ of p where U is a closed disc with N boundary points identified to a single point w. Also $\mathcal{G} \mid \mathring{U} - p$ is a foliation by circles and $\mathcal{G}$ has a saddle singularity at w of index $-M$. See Figures 7.1, 7.7 (left) and 7.10 of [G1] which show examples of center singularities of type (1,1) and type (2,1). In the notation of [G1] they are of type 1a and 2a.

Proof of Claim 5. Apply [Ro] or [Th]. Here is the idea. A small neighborhood of p is a disc D foliated by circles centered at p and each circle is homotopically trivial in its leaf. Using Reeb's stability theorem and the fact that $\mathcal{F}$ has no Reeb components, a maximal such "disc" D exists whose boundary contains saddle tangencies. By the genericity of the immersion the saddle singularities in ∂D all occur at a single point w. These are the N points identified with w. $\square$

Claim 6. If v is a type (N, M) center with $M \geq N$, then T can be homotoped to a normal transverse immersion which has fewer centers.

Proof of Claim 6. By considering the plateau P moments before it encounters the saddle one sees that the normal orientation to T at both the plateau P and the saddle w agree with the transverse orientation of $\mathcal{F}$. (Consider a cell of Δ which nontrivially intersects both P and w.) Thus the case of a type (1,1) center is the standard cancellation of a saddle with a center Compare Figures 7.7-7.8 of [G1]. The other cases are similar. $\square$

Remark 2.11. In the classical settings of [Ro], [Th] or §7 of [G1] the normal orientation of T at the plateau may agree with the transverse orientation of $\mathcal{F}$ at the plateau, while the transverse orientation of T at the saddle may disagree with the transverse orientation of $\mathcal{F}$ at the saddle. See Figures 7.6-7.13 [G1]. Remarkably in our least weight normal surface setting such a disagreement would give rise to a weight reducing homotopy of T. Rethink about Figure 2.4.

Claim 7. If v is a type $(N, N-1)$ center, then either T can be homotoped to a Γ-normal transverse immersion with fewer centers or T can be homotoped to a normal immersion which, away from a non simply connected plateau P, is a Γ-normal transverse immersion. The immersion has one fewer center (the lost center being transformed into the plateau) and one fewer saddle.

Proof of Claim 7. A plateau resulting from a $(N, N-1)$ center which has been pushed to the level of its saddle is an immersed pinched non simply connected surface P'. If $\mathcal{F}$ has holonomy below this surface, then a small homotopy of T near P' gives rise to a Γ-normal transverse immersion with fewer center tangencies. This type of operation was first noticed by Roussarie [Ro] and it is an amusing exercise for the reader. (Hint: Eliminate an annulus plateau which has nontrivial holonomy.) If $\mathcal{F}$ has no holonomy below the pinched surface, then a further down push gives rise to a plateau P which is an immersed nonsimply connected surface which has absorbed a saddle of $\mathcal{F} \mid T$. $\square$

As in [Ro] we now repeat the entire story with minor modification. We redefine type b) events as follows. "The plateau meets either a saddle point or a local minimum at a vertex of Δ." Thus a plateau P absorbes a center during a type b) event involving a local minimum. At such a moment it is possible that $P = T$ and therefore T is an immersion onto a leaf.

Again by Reeb Stability we may push down P until either it encounters events of type a), b) or c) or until P reaches a level where it has holonomy. To prove this, use the fact that P is a finite union of pieces of normal discs and each such piece has a maximal push down extension before encountering a vertex of Δ or a vertex of Γ. Piecing the local pushes together and analyzing the result gives rise to the desired conclusion.

Except for the additional possibility of P encountering a moment for which it has holonomy, Claims 2-6 are exactly analogous. As in the $P = D^2$ case, if not stopped for holonomy reasons, P eventually will encounter a type b) event at a vertex w of Δ. At a type b) event either P absorbs a center as discussed above or it encounters a saddle. If the latter happens then, in analogy to the disc case, we say that the plateau is of type $(N, M), M \geq N - 1$ and $N, M \in \mathbb{N}$ if at the level of the saddle w it is topologically a surface with N boundary points identified to w, and that $\mathcal{G}$ has a singularity of index $-M$ at w. Thus the plateau can be eliminated if is not of type $(N, N - 1)$. For example if P was pushed to the level of an index -1 saddle and $N = 1$, then a small neighborhood of P would be replaced by a surface transverse to $\mathcal{F}$ which has only saddle tangencies. Without introducing local maxima or minima a small homotopy transforms T into a Γ-normal transverse immersion.

If P is of type $(N, N - 1)$ and there is holonomy under the plateau when P has reached the level of w, then again as in [Ro] a homotopy supported in a neighborhood of the plateau transforms that neighborhood into one transverse to $\mathcal{F}$ except at saddle tangencies. If the plateau has trivial holonomy, then upon passing through the saddle w it will have absorbed a saddle of $\mathcal{G}$ and will become a new plateau of smaller Euler characteristic. Since $\mathcal{G}$ has finitely many saddle tangencies this process must terminate. This completes the proof of Theorem 2.7. $\square$

§3. Generalizations

In the theory of PL minimal surfaces [Ha], [JR], the weight assigned to each 1-simplex is 1. However it is immediate that the whole theory goes through using positive real numbers as weights. (There are interesting interpretations when some of the weights are allowed to be zero.) As we have seen, non unit weights arise naturally in the context of combinatorial volume preserving flows.

While the PL theory has greatly clarified the smooth minimal surface theory, it has the following defect. That is, a solution to the following

Problem 3.1. *Find a way to generalize the PL theory so that the space of PL metrics on a given 3-manifold is path connected. (Is contractibility too much to hope for?)*

Remarks 3.2. i) For example, the author found a PL version of Proposition 3.9 of [G4], but because of this gap could not find a suitable PL version of Proposition 3.10 of [G4] and had to work in the smooth category.

In Haken's theory [Ha], the weight of a surface in a triangualated 3-manifold is its intersection with the 1-skeleton. Thus the mass of the manifold is concentrated in the 1-skeleton. Jaco and Rubinstein [JR] refine Haken's theory by adding a second order term, which has the effect of concentrating the mass of the manifold on the 2-skeleton. Perhaps a third order term must be added to the theory in order to understand Problem 3.1.

Remark 3.3. The orientability of M and T and the transverse orientability of $\mathcal{F}$ in the statement of Theorem 2.7 were inessential. Neither were the hypothesis of M and T being closed. To deal with these cases we have the following analogue of Definition 0.1.

Definition 3.4. A *combinatorial volume preserving line field* (ϕ, Γ) on the 3-manifold M consists of a triangulation Γ and an assignment (ϕ, Γ) of a positive integer $\phi(\kappa)$ to each 1-simplex κ of Γ. Additionally, the edges of hitting each vertex v are partitioned into two sets A_v, B_v such that the sum of the ϕ-values of A_v-edges equals the sum of the values of the B_v-edges.

The line field (ϕ, Γ) is *compatible* with the foliation $\mathcal{F}$ if $\mathcal{F}$ intersects Γ normally and if $v \in \Gamma^0$, then $\sum_{i=1}^{n} \phi(\alpha_i) = \sum_{j=1}^{m} \phi(\beta_j)$. Where with respect to some local transverse orientation on $\mathcal{F}$ near v, $\{\alpha_i\}$ (resp. $\{\beta_j\}$) are the edges emanating from v which lie below (resp. above) v.

Remark 3.5. i) If $\mathcal{F}$ is transversely oriented, then a line field (ϕ, Γ) compatible with $\mathcal{F}$ is a combinatorial volume preserving flow.

ii) The restriction of (ϕ, Γ) to ∂M need not be a combinatorial volume preserving flow on ∂M.

We have the following analogue to Lemma 1.2.

Lemma 3.6. *Let $\mathcal{F}$ be a codimension-1 foliation on the compact orientable 3-manifold M such that $\mathcal{F}$ is transverse to ∂M. $\mathcal{F}$ is a taut or a nontransversely orientable taut foliation if and only if then there exists a combinatorial volume preserving line field (ϕ, Γ) compatible with $\mathcal{F}$.* □

We have the following analogue to Theorem 2.7.

Theorem 3.7. *Let $\mathcal{F}$ be either a taut or a nontransversely orientable taut foliation on the compact 3-manifold M such that $\mathcal{F}$ is transverse to ∂M. If T is a compact properly immersed incompressible and boundary incompressible surface in M such that each component of ∂T is either transverse to $\mathcal{F}|\partial M$ or tangent to $\mathcal{F}|\partial M$, then either T can be properly homotoped to an immersion into a leaf of $\mathcal{F}$ or T can be properly homotoped to an immersed surface transverse to $\mathcal{F}$ except at isolated saddle tangencies.* □

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